

Statistical physics for complex cosmic structures

Luciano Pietronero¹ and Francesco Sylos Labini²

¹ Physics Department University of Rome “La Sapienza”, P.le A. Moro 2, 00185 Rome Italy

² Université de Paris-Sud, Laboratoire de Physique Theorique Bâtiment 210, F-91405 Orsay Cedex, France

Abstract. Cosmic structures at large scales represent the earliest and most extended form of matter condensation. In this lecture we review the application of the methods and concepts of modern statistical physics to these structures. This leads to a new perspective in the field which can be tested by the many new data which are appearing in the near future. In particular, galaxy structures show fractal correlation up to the present observational limits. The cosmic microwave background radiation, which should trace the initial conditions from which these structures have emerged through gravitational dynamics, is instead extremely smooth. Understanding the relation between the complex galaxy structures and the smooth microwave background radiation represents an extremely challenging problem in the field of structure formation.

1 Introduction

The specific issue which initiated our activity at the interface between statistical physics and cosmology, was the study of the clustering properties of galaxies as revealed by large redshift surveys, a context in which concepts of modern statistical physics (e.g. scale-invariance, fractality..) found a ready application [1–3]. In recent years we have broadened considerably the range of problems in cosmology which we have addressed, treating in particular more theoretical issues about the statistical properties of standard primordial cosmological models [4–6] and the problems of formation of non-linear structures in gravitational N-body simulations [7,8]. What is common to all this activity is that it is informed by a perspective and methodology which is that of statistical physics: in fact we believe that this represents an exciting playground for statistical physics, and one which can bring new and useful insights into cosmology. In this lecture we briefly review the main points and we refer the interested reader to the recent monograph on the subject [4].

2 Galaxy structures

The question which first stimulated our interest in cosmology has been the correlation properties of the observed distribution of galaxies and galaxy clusters [1–3]. It is here that the perspective of a statistical physicist, exposed to

the developments of the last decades in the description of intrinsically irregular structures, is radically different from that of a cosmologist for whom the study of fluctuations means the study of small fluctuations *about a positive mean density*. And it is here therefore that the instruments used to describe strong irregularity, even if limited to a finite range of scales, offer a wider framework in which to approach the problem of how to characterize the correlations in galaxy distributions. Analyzing galaxy distributions in this wider context means treating these distributions without the *a priori* assumption of homogeneity, i.e. without the assumption that the finite sample considered gives, to a sufficiently good approximation, the true (non-zero) mean density of the underlying distribution of galaxies. While this is a simple and evident step for a statistical physicist, it can seem to be a radical one for a cosmologist [9,10]. After all the whole theoretical framework of cosmology (i.e. the Friedmann-Robertson-Walker — FRW — solutions of general relativity) is built on the assumption of an homogeneous and isotropic distribution of matter. Our approach is an empirical one, which surely is appropriate when faced with the characterization of data. Further it is evidently important for the formulation of theoretical explanations to understand and characterize the data correctly.

The maps probing the distribution of galaxies revealed in the first large three dimensional surveys, which were published in the eighties, structures - super-clusters, walls, voids, filaments - at scales much larger than had been suspected from the previous (fairly isotropic) angular data (see Fig.1). The simple visual impression of such three-dimensional data - apparently showing large fluctuations up to the sample sizes - gives a strong *prime face* case for an analysis which does not assume the underlying distribution is uniform (at the scales probed), but rather encompasses the possibility that it may be intrinsically irregular. The quantitative results obtained with the standard analysis performed on these samples, which works with statistics which build in the assumption of homogeneity, give further evidence in this direction.

In simple terms, the standard statistical framework used in cosmology is the one developed for the studies of regular distributions as fluids: one assumes that the average density $\langle n \rangle$ is well defined within a given sample and thus one focuses on the description of fluctuations around such a quantity (where $\langle \dots \rangle$ is the unconditional ensemble average). In order to use a normalized (or reduced) two-point correlation function one defines

$$\xi(r) = \frac{\langle n(r)n(0) \rangle}{\langle n \rangle^2} - 1 \quad (1)$$

so that $\xi(r) \neq 0$ implies the presence of correlations, while $\xi(r) = 0$ is for a purely Poisson (uncorrelated) distribution: when $\xi(r) \gg 1$ the regime is of strong-clustering while $\xi(r) < 0$ describes anti-correlations. However such a definition does not include the possibility that the average density $\langle n \rangle$ could

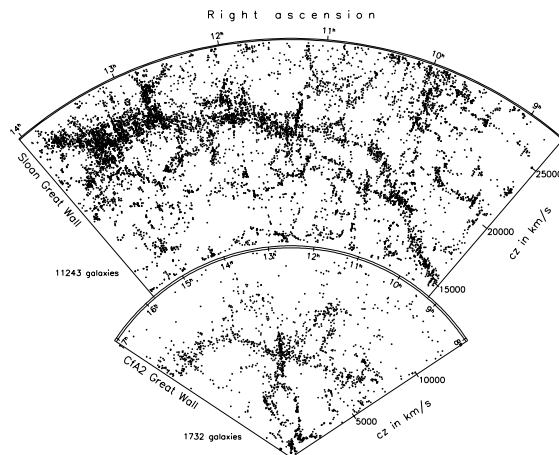


Fig. 1. Sloan Great Wall compared to CfA2 Great Wall at the same scale.. Redshift distances cz are indicated (distances are simply given by $cz/100$ Mpc/h). The Sloan slice is 4 degrees wide, the CfA2 slice is 12 degrees wide to make both slices approximately the same physical width at the two walls. (From [12] - see also <http://www.astro.princeton.edu/~mjuric/universe/>)

not be a well-defined quantity, i.e. that it could be a function of the sample size. This can be considered by using the conditional average density defined as

$$\langle n(r) \rangle_p = \frac{\langle n(r)n(0) \rangle}{\langle n \rangle} \quad (2)$$

which gives the average density around an occupied point of the distribution [1,2,4] (where $\langle \dots \rangle_p$ is the conditional ensemble average). Clearly there is a simple relation which links these two functions

$$\langle n(r) \rangle_p = \langle n \rangle (1 + \xi(r)) ,$$

which is physically meaningful only in the case when the average density is a well defined quantity. Together with this simple change of statistical description of correlations one should consider however a subtle but important point related to the non-analytical character of fractal objects.

Fractal geometry, has allowed us to classify and study a large variety of structures in nature which are intrinsically irregular and self-similar [13]. The *metric* dimension is the most important concept introduced to describe these

intrinsically irregular systems. Basically it measures the rate of increase of the “mass” of the set with the size of the volume in which it is measured: In terms of average density around an occupied point (averaged over all occupied points) if it scales as

$$\langle n(r) \rangle_p \sim r^{D-3} \quad (3)$$

then the exponent D is called the *mass-length dimension* and it is $0 < D \leq 3$ (in the three dimensional Euclidean space, where $D = 3$ corresponds to the uniform case). In this situation the reduced correlation function (Eq.1) loses its physical meaning although its estimator can be defined in a finite sample. This is so, because its amplitude depends on the sample size as the average density does. In fact one may show that in this case the estimator of $\xi(r)$ in a finite spherical sample of size R_s becomes (in an infinite volume it cannot be defined as a fractal is asymptotically empty, i.e. $\langle n \rangle = 0$)

$$\xi_E(r) = \frac{D}{3} \left(\frac{r}{R_s} \right)^{D-3} - 1. \quad (4)$$

The use of the conditional average density we consider (Eqs.2-3) encompasses both some irregular distributions (simple fractals) and uniform distributions with small scale clusterization, in particular those described by standard cosmological models. If the sample size is much larger than the homogeneity scale λ_0 one can detect using $\langle n(r) \rangle_p$ the existence and location of λ_0 . If, on the other hand the sample’s size is smaller than λ_0 the conditional density allows the determination of the fractal exponent characterizing the clustering on these scales.

While the reduced correlation function $\xi(r)$ has been found to show consistently a simple power-law behavior characterized by the same exponent in the regime of strong clustering, there is very considerable variation between samples, with different depth and luminosity cuts, in the measured *amplitude* of mean-density-normalized correlation function [14]. This variation in amplitude is usually ascribed *a posteriori* to an intrinsic difference in the correlation properties of galaxies of different luminosity (“luminosity bias”, or “luminosity segregation”). It may, however, have a much simpler explanation in the context of irregular distributions. As mentioned above, in a simple fractal (at least within the sample scale), for example, the density in a finite sample decreases on average as a function of sample size; samples of increasingly bright galaxies are in fact generically of greater mean depth, which corresponds to an increasing amplitude of the correlation function $\xi(r)$ normalized to the “apparent” average density in each sample (see Eq.4).

With this motivation we have applied these simple statistical methods which allow a characterization of galaxy clustering, irrespective of whether the underlying galaxy distribution is homogeneous or irregular at the sample size. Up to scales of tens of Mpc, we have found that galaxy distributions in many different surveys [2,3] show fractal properties corresponding to $D \approx 2$

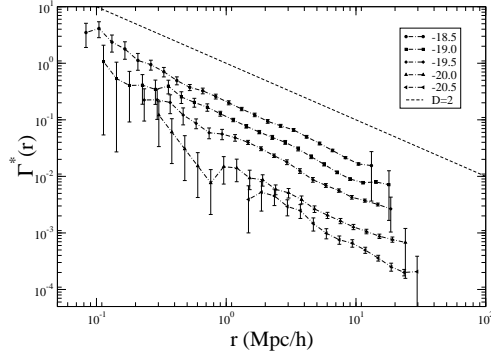


Fig. 2. The estimator of the conditional average density $\Gamma^*(r)$ in the different samples of the CfA2-South galaxy catalog. (Adapted from [3]).

with no robust evidence for homogeneity (see Fig.2). It is on this important issue that future surveys (e.g. the Sloan Digital Sky Survey — SDSS — [11]) will allow us to place much tighter constraints. We stress however the following important points: if the homogeneity scale is determined to be finite, the fractal-inspired analysis remains valid in two respects: (i) it is the unique way to detect the homogeneity scale itself without a priori assumptions, and (ii) it gives the right framework to obtain a full geometrical and statistical description of the strongly clustered region [4]. This result has caused a passionate debate in the field because it is in contrast with the usual assumption of large scale homogeneity which is at the basis of most theories. Our basic conclusion is that, given the fractal nature of galaxy clustering on small scales, one should change the general perspective with respect to the problem of non-linear structure formation. Note that this is the case even it turns out that there is crossover to homogeneity because, at least in a certain range of scales, that in which there is strong clustering, structures have a fractal nature and this cannot be described as perturbations to a smooth fluid. It can be interesting to note that some cosmologists (e.g. [9]) have confirmed our measurements about fractality at scales of order tens Mpc, the debate about the transition to homogeneity concerning scales of one order of magnitude larger. The discussion about the homogeneity scale is indeed an hot topic in contemporary science, touching many deep and important key-issues in different fields of physics, from large scale astrophysics to gravitation, to the question of the nature and amount of dark matter. To these topics it is dedicated a recent book [16]: Its main aim is in fact to address, in a novel and open way, the problem of structures in cosmology, with a reach historical perspective and stressing the open questions which arise from the consideration of their intrinsic complexity.

One of the reason of the resistance to our ideas in cosmology, is that we do not present neither an alternative theory or a model nor a physical theory

for the fractal growth phenomena in the gravitational case: this is due to the fact that this issue is indeed extremely difficult as any out-of-equilibrium irreversible dynamical growth process. Actually the study of such processes is at the frontier of the theoretical efforts in statistical physics. Let us, however, note briefly two common *theoretical* objections to this approach. One is that fractals are incompatible with what is called the “cosmological principle”, by which it is meant that there is no privileged point or direction in the universe. This is a simple misconception about fractals. These irregular distributions are in keeping with this principle in exactly the way as inhomogeneities treated in the standard framework of perturbed FRW models, i.e. they are *statistically* stationary and isotropic distributions in space in the sense that their statistical properties are invariant under translation and rotation in space. Another common objection is that there is an inconsistency in using the Hubble law, which is used to convert redshift to physical distance, if one does not assume homogeneity. This objection forgets that the Hubble law is an established empirical relation, independent of theories explaining it. Further it can be noted that what we are concerned with is the distribution of *visible* matter: given that standard cosmological models describe a universe whose energy density is completely dominated by several non-visible components, reconciling the two is not in principle impossible [15].

3 Primordial density fields

As already mentioned, the standard interpretation of galaxy and cluster correlations results from the $\xi(r)$ analysis and can be summarized as follows: (i) Power-law behavior in the regime of strong clustering ($\xi \gg 1$) (ii) Amplitude which changes from sample to sample. These observations of the varying amplitude are ascribed to real physics, rather than being simply finite size effects as in the alternative explanation we present. Generically this goes under the name of “bias”, which is then simply taken to mean any difference between the correlation properties of any class of object (galaxy of different luminosity or morphology, galaxy cluster, quasar...) and that of the “cold” dark matter (CDM) which, in standard models, dominates the gravitational clustering dynamics in the universe. The case that such a difference manifests itself as an overall normalization of $\xi(r)$, as in the observations we have been discussing, is known as “linear bias”. In order to understand the problematic aspects of this interpretation, one should consider an important feature common to all standard theoretical models of cosmological density fields, like CDM.

First let us briefly summarize the main properties of these fields. Matter distribution in the early universe is supposed to be well represented by a continuous Gaussian field with appropriate correlation properties. According to the standard picture of the Hot-Big-Bang scenario [10] matter density fluctuations had interact with radiation at early times leaving therefore an imprint

of its correlation properties of the cosmic microwave background radiation (CMBR) anisotropies, in particular determining an important property at large scales. This is the so-called Harrison-Zeldovich (HZ) condition on the power spectrum (PS) $P(k)$ (Fourier Transform of $\xi(r)$) of these tiny fluctuations with respect to perfect uniformity [6]: $P(k) \sim k$ at small k (large scales). We have noticed in [6] that this condition implies that their primary characteristic is to show surface mass fluctuations, which are the most depressed fluctuations possible for any stochastic distribution. In a simple classification of all stationary stochastic processes into three categories, we highlighted with the name “super-homogeneous” the properties of the class to which models like this belong: let us briefly discuss this point.

A simple description of these systems can be the following. An uncorrelated Poisson system (or with positive and finite range of correlations) has at large scale $\xi(r) = 0$ and hence the PS is $P(k) \sim \text{const}$: for this reason it satisfies the condition

$$\int_0^\infty \xi(r)r^2 dr \sim P(0) = \text{const}.$$

In this situation mass fluctuations in spheres grows as $\langle \Delta M^2(R) \rangle \sim R^3$ (Poisson fluctuations): this is, for example a perfect gas at thermodynamical equilibrium. On the other hand, one may find systems with a power-law correlation function as in thermodynamical critical phenomena. In this situation $\xi(r) \sim r^{-\gamma}$ where $0 < \gamma < 3$, $P(k) \sim k^{-(3-\gamma)}$ and

$$\int_0^\infty \xi(r)r^2 dr \sim P(0) = \infty :$$

The correlation length diverges and the system has critical features is that mass fluctuations growth faster than in the Poisson case, i.e. $\langle \Delta M^2(R) \rangle \sim R^{6-\gamma}$. Finally there are systems such that

$$\int_0^\infty \xi(r)r^2 dr \sim P(0) = 0 \tag{5}$$

and mass fluctuations in spheres go as $\langle \Delta M^2(R) \rangle \sim R^2$: In statistical physics language they are well described as glass-like or long-range ordered configurations. For example the generation of points distributions in three dimensions with properties similar to HZ ones encountered in statistical physics is represented by the one-component system of charged particles in a uniform background [5]. This latter system, appropriately modified, can produce equilibrium correlations of the kind assumed in the cosmological context. These systems are characterized by a long-range balance between correlations and anti-correlations (Eq.5) and show the most depressed possible fluctuations in any stochastic distribution [6].

The HZ type spectrum was first given a special importance in cosmology with arguments for its “naturalness” as an initial condition for fluctuations

in the framework of the expanding universe cosmology. Basically it satisfies the global condition on large scale mass fluctuations $P(0) = 0$, which is the only one compatible with the FRW metric, as it avoids the divergence of the fluctuations of the gravitational potential (as it happens in the simple Poisson case [6]). It subsequently gained in importance with the advent of inflationary models in the eighties, and the demonstration that such models quite generically predict a spectrum of fluctuations of this type. Since the early nineties, when the COBE experiment (and then more recently WMAP [17]) measured for the first time fluctuations in the temperature CMBR at large angular separations, and found results consistent with the predictions of models with a HZ spectrum at such scales, the HZ type spectra have become a central pillar of standard models of structure formation in the universe.

Because of the highly irregular nature of structure at small scales, standard models with super-homogeneous features cannot be used even at zeroth order to describe these observed structures. This does not mean, however, that these models cannot describe successfully galaxy structures: but to establish whether they can, it must first be shown from observations that there is a clear crossover toward homogeneity i.e. a scale beyond which the average density becomes a well-defined (i.e. sample-independent) positive quantity. These models then predict that, on much larger scales (e.g. > 100 Mpc), galaxy structures should present the super-homogeneous character of the HZ type PS. Indeed this should in principle be a critical test of the paradigm linking the measurements of CMBR on large scales to the distribution of matter. Observationally a crucial question is the feasibility of measuring the transition between these regimes directly in galaxy distributions. In this context one has to consider an important element: the galaxy distribution is a discrete set of objects whose properties are related in a non-trivial way to the ones of the underlying continuous field. To understand the relation between the two, one has to consider the additional effects related to sampling the continuous field. This is intimately related to the problem of “biasing” between the distribution of visible and dark matter which, as above mentioned, is usually invoked to explain a posteriori the observational features of $\xi(r)$.

Sampling a super-homogeneous fluctuation field may change the nature of correlations [18]. The reason can be found in the property of super-homogeneity of such distribution: the sampling, as for instance in the so-called “bias model” (selection of highest peaks of the fluctuations field) necessarily destroys the surface nature of the fluctuations, as it introduces a volume (Poisson-like) term in the variance. The “primordial” form of the power spectrum is thus not apparent in that which one would expect to measure from objects selected in this way. This conclusion should hold for any generic model of bias [18]. If a linear amplification is obtained in some regime of scales (as it can be in certain phenomenological models of bias) it is necessarily a result of a fine-tuning of the model parameters. The study of different samplings and of the correlation features invariant under sampling represents an impor-

tant issue in relation to the comparison of observations of galaxy structures, or distributions given by N-body simulations with primordial fluctuations (CMBR anisotropies) and theoretical models (see Fig.3).

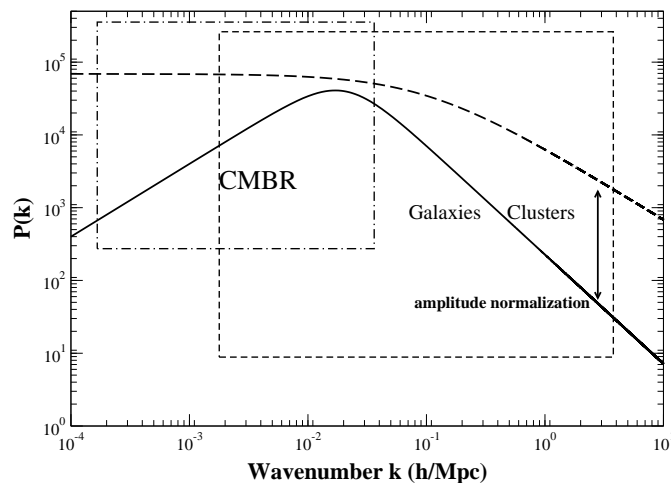


Fig. 3. In this figure we show a typical CDM type power spectrum (solid line) and the range associated to the primary observational constraints. The left hand side, the Harrison Zeldovich part of the PS ($P(k) \sim k$), is constrained by observations of the anisotropies of the CMBR (dash-dotted box). Current galaxy and galaxy cluster surveys gives constraints at smaller scales (dashed box). The normalization of the amplitude of the galaxy or cluster PS to the one observed in the CMBR is fundamentally important. It is usually determined by a *linear* re-scaling on the y-axis, ascribed to the effect of bias. This simple assumption is not consistent with the canonical model for the biasing of a Gaussian field, which introduces a non-linear distortion both at small and large wave-number. This is illustrated by the dashed line, which shows what is actually obtained for the PS of the biased field. On small scales (large k) there is a non-linear distortion and at large scales (small k) the behavior is typical of a substantially Poisson system with $P(k) \sim \text{const.}$ (From [4]).

4 The problem of gravitational structure formation

In cosmology the main instrument for treating the *theoretical* problem of gravitational clustering is numerical (in the form of N-body simulations —NBS—), and the analytic understanding of this crucial problem is very limited. Other than in the regime of very small fluctuations where a linear analysis can be performed, the available models of clustering are essentially phenomenological models with numerous parameters which are fixed by numerical simulation. An NBS has these main features: (i) given N particles in a volume

V , each particle moves according to the force given by the sum of the other $N - 1$ particles and of all their replicas (due to the use of periodic boundary conditions). (ii) The equation of motions are integrated by using a leap-frog algorithm. (iii) Suitable techniques allow an appropriate summations (which makes the code faster) of far away particles contributions to the gravitational force. (iv) One may, or may not, consider the presence of space expansion. A typical result of the non-linear evolution is shown in Fig.4.

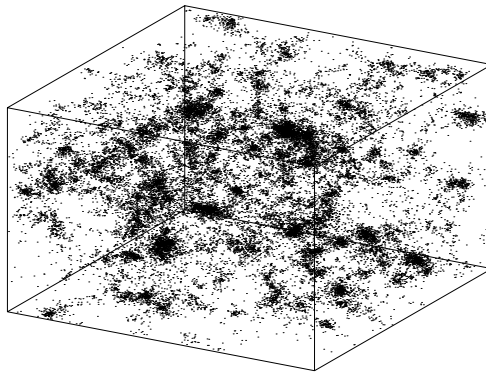


Fig. 4. This is a distribution obtained by starting from a Poisson particle configuration with zero initial velocity dispersion and by evolving it with gravity and by considering periodic boundary conditions. One may see that non-linear strong clustering is formed and its explanation is the central task of theoretical modeling. (Courtesy of T. Baertschiger).

A central issue in the context of cosmological NBS is to relate the formation of non linear structures to the specific choice of initial conditions used: this is done in order to constrain models with the observations of CMBR anisotropies, which are related to the initial conditions, and of galaxy structures, which give instead the final configuration of clustered matter. Standard primordial cosmological theoretical density fields, like the CDM case, are Gaussian and made of a huge number of very small mass particles, which are usually treated theoretically as a self-gravitating collision-less fluid: this means that the fluid must be dissipation-less and that two-body scattering should be negligible. The problem then being in which limit NBS, based on particle dynamics, are able to reproduce the two above conditions [7,19]. In this context one has to consider the issue of the physical role of particle fluctuations in the dynamics of NBS. In fact, in the discretization of a continuous density field one faces two important limitations corresponding to the new length scales which are introduced. On the one hand a relatively small num-

ber of particles are used. This introduces a mass scale which is the mass of these particles. In typical cosmological NBS, this mass is of the order of a galaxy and hence many orders of magnitude larger than the microscopic mass of a CDM particle. Furthermore, it introduces a new length scale given by the average distance between nearest neighbor particles. On the other hand one must regularize the gravitational force at small scales in order to avoid numerical problems and typical small scale effects due to the discrete nature of the particles: given that the smoothening length is typically smaller than the initial inter-particle separation, it is not evident that one is effectively reproducing a dynamics where particles play the role of collision-less fluid elements. It is in this sense that one talks about the role of discreteness in NBS: that strong scattering between nearby particles are produced by the discretization and they should be considered artificial and spurious with respect to the dynamical evolution of a self-gravitating fluid. This point has been considered by many authors and they all show that discreteness has some influence on the formation of the structures [20,19]. Indeed, discreteness may play an important role in the early times formation of non-linear structures. How discrete effects are then “exported” toward large scales, if they are at all, is then an important but difficult problem to be understood. In other words the problem is that of understanding whether large non-linear structures, which at late times contain many particles, are produced solely by collision-less fluid dynamics, or whether the particle collisional processes are important also in the long-term, or whether they are made by a mix of these two effects [19].

For example in [8] we have presented an analysis of different sets of gravitational NBS all describing the dynamics of discrete particles with a small initial velocity dispersion. They encompass very different initial particle configurations, different numerical algorithms for the computation of the force, with or without the space expansion of cosmological models. Despite these differences we find in all cases that the non-linear clustering which results is essentially the same, with a well-defined simple power-law behavior in the conditional density in the range from a few times the lower cut-off in the gravitational force to the scale at which fluctuations are of order one. We have argued, presenting quantitative evidence, that this apparently universal behavior can be understood by the domination of the small scale contribution to the gravitational force, coming initially from nearest neighbor particles [19]. A more quantitative description of this dynamics is evidently needed, with the principal goal of understanding the specific value observed of the exponent. In the cosmological literature (see e.g. [10]) the idea is widely dispersed that the exponents in non-linear clustering are related to that of the initial PS of the small fluctuations in the CDM fluid, and even that the non-linear two-point correlation can be considered an analytic function of the initial two-point correlations. The models used to explain the behavior in the non-linear regime usually involve both the expansion of the Universe, and a description

of the clustering in terms of the evolution of a continuous fluid. We have argued that the exponent is universal in a very wide sense, being common to the non-linear clustering observed in the non-expanding case. It would appear that the framework for understanding the non-linear clustering must be one in which discreteness (and hence intrinsically non-analytical behavior of the density field) is central, and that the simple context of non-expanding models should be sufficient to elucidate the essential physics.

There are then some basic questions which remain unanswered: does gravitational dynamics give rise to fractal clustering ? And, in case, from what initial conditions ? We believe there is much unexplored space for a statistical physics approach to the problem of gravitational clustering, which should be able to shed light on the more general characteristics of non-linear structure formation.

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